Test 3 - MTH 2410

Dr. Graham-Squire, Fall 2012

8:11

Name: Key

I pledge that I have neither given nor received any unauthorized assistance on this exam.

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(signature)

DIRECTIONS

1. Show all of your work and use correct notation. A correct answer with insufficient work or incorrect notation will lose points.
2. Clearly indicate your answer by putting a box around it.
3. Computers are allowed on one part of this test, the very last question. Calculators are allowed on all other parts of the test. Even on questions where technology is allowed, you should still show all of your work.
4. Give all answers in exact form, not decimal form (that is, put π instead of 3.1415, $\sqrt{2}$ instead of 1.414, etc) unless otherwise stated.
5. Make sure you sign the pledge.
6. Number of questions = 9. Total Points = 90.

1. (8 points) TRUE OR FALSE. Circle the correct answer. If false, give a counterexample or explain (briefly) why it is false. If true, no explanation is necessary (though if you are wrong, an explanation can get you some partial credit).

(a) **True or False:** $\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx$. Note: a, b, c and d may be either constants or functions.

False. If c and/or d are functions of y, then $\int_c^d \int_a^b f(x,y)\,dy\,dx$ would not make sense because they would need to be constants

(b) **True or False:** If $f(r,\theta)$ is a constant function and the area of a region S is twice the area of a region R, then

$$2\iint_R f(r,\theta)dA = \iint_S f(r,\theta)dA$$

True! If f is constant, say $f(r,\theta)=k$,

then $2\iint_R f(r,\theta)\,dA = 2(\text{area of } R)\cdot k$

$= (\text{area of } S)\cdot k$

$= \iint_S f(r,\theta)\,dA$

(c) **True or False:** The method of Lagrange Multipliers always finds an absolute maximum and minimum for a given situation.

It may only find a max or a min, or may find nothing at all.

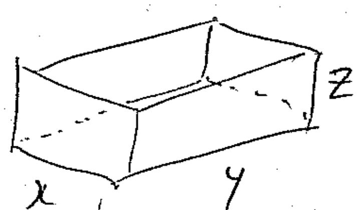
2. (12 points) A cargo container (in the shape of a rectangular solid) must have a volume of 600 ft^3. The bottom will cost \$5 per square foot and the sides and top will cost \$3 per square foot.

(a) Draw a diagram of the situation.

(b) Set up equations for the volume of the container and the cost to construct the container.

(c) Use any strategy of optimization (Lagrange multipliers or substitution) to find the dimensions of the box which will give a *minimum* cost to construct the container. You do <u>not</u> need to check that your solution is a minimum. (Round answer to nearest 0.01 ft)

(a)



(b) $xyz = 600$, $C(x,y,z) = 5xy + 3(2yz + 2xz + xy)$

$C(x,y,z) = 8xy + 6yz + 6xz$

$xyz = g = 600$

$g_x = yz$ $\quad C_x = 8y + 6z$

$g_y = xz$ $\quad C_y = 8x + 6z$

$g_z = xy$ $\quad C_z = 6y + 6z$

$\Rightarrow yz = \lambda(8y + 6z)$, $xz = \lambda(8x + 6z)$, $xy = \lambda(6y + 6z)$

$\Rightarrow \dfrac{y\cancel{z}}{8y+6z} = \dfrac{x\cancel{z}}{8x+6z} \Rightarrow \cancel{8xy} + 6zy = \cancel{8xy} + 6zx$

$\Rightarrow x = y$

$\Rightarrow x^2 z = y^2 z = 600$

$\dfrac{\cancel{x}z}{8x+6z} = \dfrac{\cancel{x}y}{6y+6z} \Rightarrow z(6x+6z) = x(8x+6z)$

$\cancel{6zx} + 6z^2 = 8x^2 + \cancel{6zx}$

$z^2 = \frac{8}{6}x^2$

$\Rightarrow z = \frac{2}{\sqrt{6}}x$

$\Rightarrow x^3 \cdot \frac{2}{\sqrt{6}} = 600$

$x^3 = \dfrac{600\sqrt{6}}{2}$

$\Rightarrow x = \sqrt[3]{\dfrac{600\sqrt{6}}{2}} \approx 9.02$

$y = 9.02$

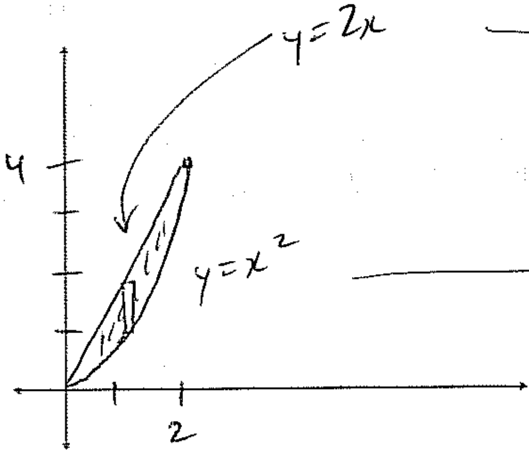
$z = 7.37$

2 ~~3~~. (10 points) (a) Sketch the region R that is bounded by $y = 2x$ and $y = x^2$.

(b) Set up a double integral to represent the area of the region R. Then change the limits of integration to represent R by a different double integral.

(c) Evaluate whichever integral you believe to be easier.

(a)



$y=2x \longrightarrow x = \frac{1}{2}y$

$y=x^2 \longrightarrow x = \sqrt{y}$

(b) Area $= \displaystyle\int_0^2 \int_{x^2}^{2x} dy\, dx$ or $\displaystyle\int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} dx\, dy$

(c)
$$= \int_0^2 (2x - x^2)\, dx$$
$$= x^2 - \frac{1}{3}x^3 \Big|_0^2$$
$$= 4 - \frac{8}{3} = \boxed{\frac{4}{3}}$$

For questions ~~4~~ 3 to ~~9~~ 7, use any technique of integration to answer the question. It may be helpful at times to change the limits of integration and/or change to a different coordinate system.

Set up but do not integrate.

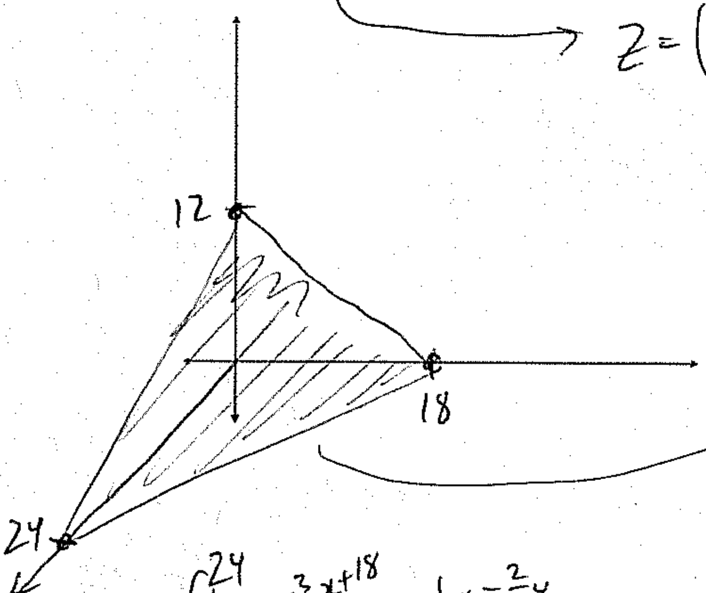
8:11
8:27
16 min

8:39
9:03
24

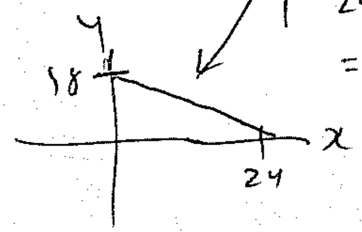
40 min

~~4~~ 3. (10 points) Find the volume of the solid formed by the intersection of the planes $3x + 4y + 6z = 72$, $x = 0$, $y = 0$, and $z = 0$.

$$z = (72 - 3x - 4y)\frac{1}{6}$$



$$y = \frac{-18}{24}x + 18$$
$$= \frac{-3}{4}x + 18$$



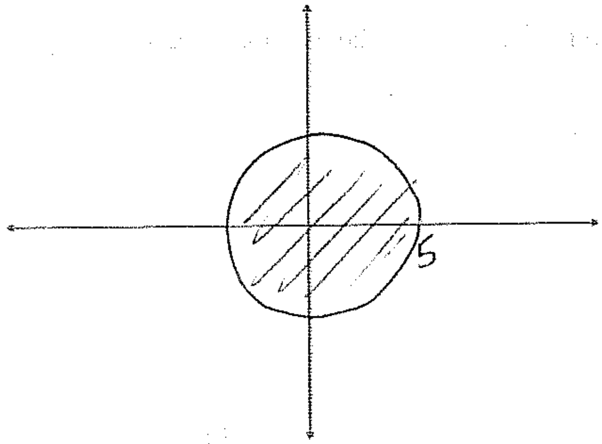
$$\int_0^{24}\int_0^{-\frac{3}{4}x+18}\int_0^{12-\frac{1}{2}x-\frac{2}{3}y} dz\,dy\,dx$$

$$= \int_0^{24}\int_0^{-\frac{3}{4}x+18}\left(12 - \frac{1}{2}x - \frac{2}{3}y\right)dy\,dx$$

$$= \int_0^{24} 12y - \frac{1}{2}xy - \frac{y^2}{3}\Big|_0^{-\frac{3}{4}x+18} dx$$ blah

4

5. (10 points) Find the volume of the solid bounded above by $f(x, y) = e^{-(x^2+y^2)/2}$, below by $z = 0$, and inside the cylinder $x^2 + y^2 = 25$.



$e^{-(x^2+y^2)/2}$

$\to = e^{-r^2/2}$

do it in polar $0 \le r \le 5$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi} \int_0^5 e^{-r^2/2} \cdot r \, dr \, d\theta$$

$u = \frac{-r^2}{2}$ $\quad 0 \to 0$, $5 \to \frac{-25}{2}$

$du = -r\,dr$

$$= -\int_0^{2\pi} \int_0^{-25/2} e^u \, du \, d\theta$$

$$= -\int_0^{2\pi} e^u \Big|_0^{-25/2} d\theta$$

$$= -2\pi\left(e^{-25/2} - e^0\right) = \boxed{2\pi\left(1 - e^{-25/2}\right)}$$

5

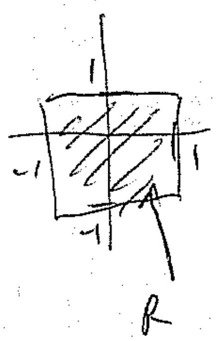
~~6.~~ (10 points) (a) Set up, but do not integrate, an integral that gives the surface area on the graph of $f(x, y) = e^{-x} \sin y$, lying above the square region bounded by $x = 1$, $x = -1$, $y = 1$ and $y = -1$.

(b) Simplify the integrand as much as possible and state if you could integrate it by hand or not, and if not, why.

$$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$f_x = -e^{-x} \sin y$$

$$f_y = e^{-x} \cos y$$



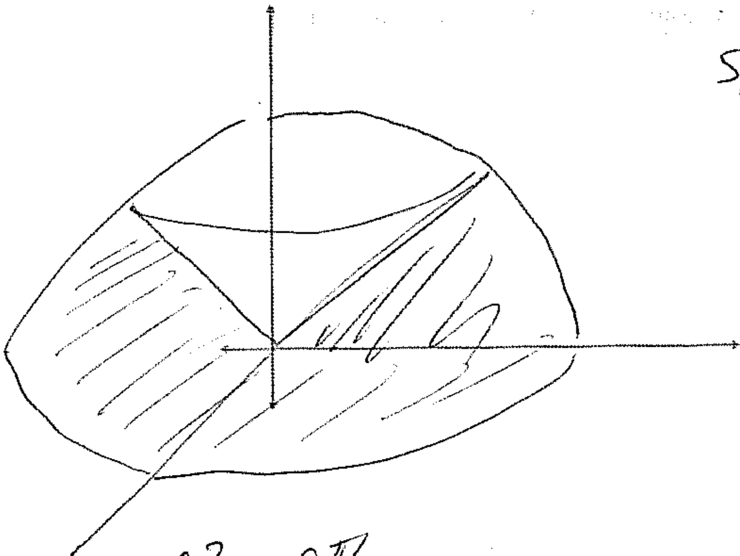
(a) $$= \int_{-1}^{1} \int_{-1}^{1} \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y}\; dx\, dy$$

(b) $$= \int_{-1}^{1} \int_{-1}^{1} \sqrt{1 + e^{-2x}}\; dx\, dy$$

Cannot integrate by hand b/c if let $1 + e^{-2x} = u$, will need a e^{-2x} term outside the $\sqrt{\;}$.

7. (10 points) Find the volume of the solid inside $x^2+y^2+z^2=16$, outside $z=\sqrt{x^2+y^2}$, and above the xy-plane.

6



Spherical: $0 \le \rho \le 4$

$\pi/4 \le \phi \le \pi/2$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \tfrac{1}{3}\rho^3\Big|_0^4 \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{64}{3} \sin\phi \, d\phi \, d\theta$$

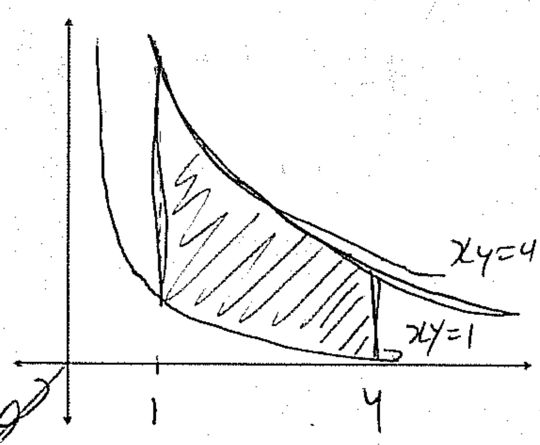
$$= \frac{64}{3}\cdot(2\pi)\left(-\cos\phi\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{128\pi}{3}\left(-0-\left(-\frac{\sqrt{2}}{2}\right)\right) = \boxed{\frac{64\pi\sqrt{2}}{3}}$$

~~8~~. (10 points) Find the volume of the solid region lying below $f(x,y) = \dfrac{xy}{1+x^2y^2}$ and above the region R bounded by the graphs of $xy = 1$, $xy = 4$, $x = 1$ and $x = 4$. (Hint: it may help to do a change of coordinates. Perhaps with $x = u$ and $y = v/u$.)

7

Round to nearest 0.01?



$xy = v \Rightarrow 1 \le v \le 4$

$1 \le u \le 4$

$\dfrac{\partial x}{\partial u} = 1 \quad \dfrac{\partial x}{\partial v} = 0$

$\dfrac{\partial y}{\partial u} = \dfrac{-v}{u^2} \quad \dfrac{\partial y}{\partial v} = \dfrac{1}{u}$

$\Rightarrow$ Jacobian $= \dfrac{1}{u}$

$$\int_1^4 \int_1^4 \frac{v}{1+v^2} \cdot \frac{1}{u}\, du\, dv$$

$$= \int_1^4 \frac{v}{1+v^2}\left(\ln|u|\Big|_1^4\right) dv$$

$$= \frac{1}{2}\ln(1+v^2)\Big|_1^4 (\ln 4 - \ln 1)$$

$$= \frac{1}{2}(\ln 5 - \ln 2)(\ln 4)$$

$$= \boxed{\left(\frac{1}{2}\ln 2.5\right)(\ln 4)} = \boxed{0.64}$$

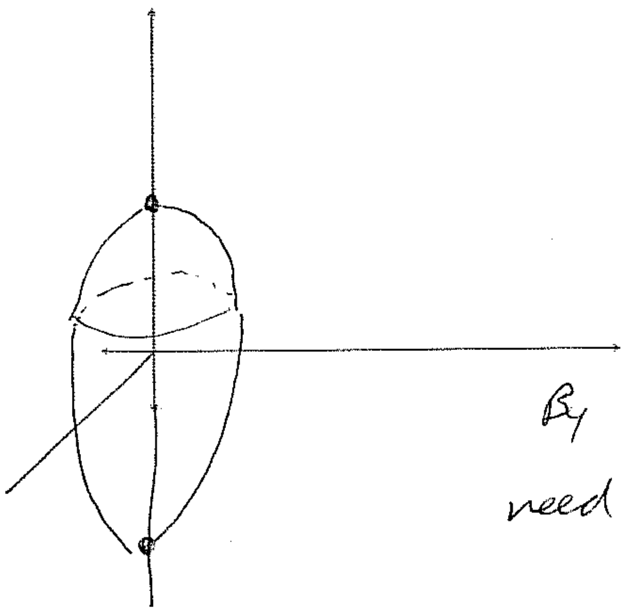
Computer allowed Name: Key

9. (10 points) Find the center of mass of a solid bounded by the paraboloids $z = 9 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 16$, with density function $\rho(x) = \sqrt{x^2 + y^2}$. You should set up the integral in the simplest coordinate system possible, then you can use a computer to find your answer(s).

$$9 - x^2 - y^2 = 3x^2 + 3y^2 - 16$$

$$25 = 4x^2 + 4y^2$$

$$\left(\frac{5}{2}\right)^2 = x^2 + y^2$$



By symmetry, $\bar{x} = 0$, $\bar{y} = 0$, just need $\bar{z} = \dfrac{M_{xy}}{m}$

in polar, $0 \le r \le \frac{5}{2}$, $0 \le \theta \le 2\pi$

$3r^2 - 16 \le z \le 9 - r^2$

$$m = \int_0^{2\pi} \int_0^{5/2} \int_{3r^2-16}^{9-r^2} \sqrt{r^2}\, r \, dz\, dr\, d\theta = 104.1\overline{6}\,\pi$$

$$M_{xy} = \int_0^{2\pi} \int_0^{5/2} \int_{3r^2-16}^{9-r^2} z\, r^2 \, dz\, dr\, d\theta = -85.5655\,\pi$$

$$\Rightarrow \frac{M_{xy}}{m} = \frac{104.1\overline{6}\,\pi}{-85.5655\,\pi} = -1.22$$

$\Rightarrow$ Center of mass is $\boxed{(0, 0, -1.22)}$

8. (12 points) A rectangular box is resting on the xy-plane with one vertex at the origin. The opposite vertex lies in the plane $6x+4y+3z=24$. Use any strategy of optimization (Lagrange multipliers or substitution) to find the dimensions of the box which will maximize the volume. You do <u>not</u> need to check that your solution is a maximum.

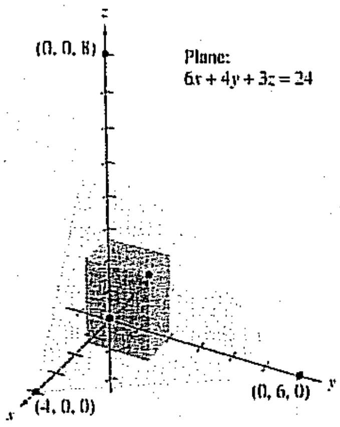


Figure 13.73

Volume of box $= xyz$

$6x+4y+3z=24$

$z=(24-6x-4y)\frac{1}{3}$

$\Rightarrow V(x,y) = 8xy - 2x^2y - \frac{4}{3}xy^2$

$V_x = 8y - 4xy - \frac{4}{3}y^2$

$V_y = 8x - 2x^2 - \frac{8}{3}xy$

critical point where $V_x = 0$, $V_y = 0$

$\Rightarrow 0 = y(8 - 4x - \frac{4}{3}y)$ $\qquad 0 = x(8 - 2x - \frac{8}{3}y)$

$x=0$ and $y=0$ are extraneous

$\Rightarrow 8 = 4x + \frac{4}{3}y$ and $8 = 2x + \frac{8}{3}y$

$\Rightarrow -16 = -4x - \frac{16}{3}y$

add together

$\Rightarrow -8 = \frac{-12}{3}y \Rightarrow \boxed{y=2}$ $\Rightarrow 8 - \frac{8}{3} = 4x$

$\frac{16}{3}\cdot\frac{1}{4} = x$

$\boxed{x = \frac{4}{3}}$

$\Rightarrow z = (24 - 6(\frac{4}{3}) - 4(2))\frac{1}{3}$

$\boxed{z = \frac{8}{3}}$